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RESEARCH ARTICLE

ON NUMERICAL INTEGRATION OF EQUALLY SPACED HYBRID METHOD FOR THIRD-ORDER INITIAL VALUE PROBLEMS USING ORTHOGONAL TRIAL FUNCTION

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ABSTRACT

The study addresses the essential role of differential equations (DEs) in modelling physical phenomena and acknowledges the challenge posed by the inability to solve many DEs analytically. To overcome this, efficient numerical and approximation methods are necessary. The focus is on constructing a family of orthogonal polynomials valid in the interval $[-1, 1]$ with a specific weight function. The hybrid two-step equally spaced method (HTEPM), employs collocation and interpolation techniques. On investigation of the fundamental properties of the method, findings reveal that the proposed schemes are consistent, zero-stable, and consequently convergent. Upon implementation, the study establishes the numerical superiority of the HTEPM over existing methods through rigorous numerical evaluations and comparisons. This suggests that the proposed method offers improved performance in solving DEs within the specified context

KEYWORDS

Collocation; Interpolation; Orthogonal Polynomials, Block Method, DEs.

1.0 INTRODUCTION

Numerical integration techniques play a pivotal role in solving differential equations, especially in scenarios where analytical solutions are unattainable. This study focuses on the numerical integration of third-order initial value problems (IVPs) utilizing an equally spaced hybrid method (HTEPM), augmented with orthogonal trial functions via collocation and interpolation technique.

Our main goal is to derive a new class of polynomials that may be used to a wide range of situations. Several writers have proposed ways for handling initial value problems because they want to improve the accuracy and efficiency of numerical approaches ([24],[26],[30]). Our goal is to

create a class of orthogonal polynomials in this work that will be used as trial functions to construct numerical methods for a class of initial value issues that look like this:

$$y^m(x) = f(x, y, y', \dots, y^{m-1}) \quad (1)$$

$$y^r(x_0) = y_r, r = 0, 1, \dots, k - 1$$

Specifically, we considered the case $m = 3$,

The analytical solution of many of such problems does not exist. Thus, the need for formulation of numerical scheme to integrate (1) becomes necessary.

Recently, there has been a focus on exploring the numerical solution of Ordinary Differential Equations (ODEs) (1) for cases where m

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equals 1, 2, and 3 using collocation methods, as evidenced by studies referenced in [1], [3], [4], [5], and [18].

More recently, [8], [9], [14], and [19] have developed various numerical methods and explored diverse trial functions, contributing to the ongoing advancement of the field

Authors in both references [14] and [21] embraced the self-starting approach, employing Chebyshev Polynomials to formulate a series of algorithms. The numerical solutions derived through their methodologies are noteworthy, as they frequently converge to exact solutions at numerous instances. In what follows, we shall construct a set of orthogonal polynomials valid in interval [0, 1] with respect to weight function $w(x) = 1 + \frac{x}{2}$ which are serve as trial functions to derive a block method that provides direct solution to (1).

2.0 CONSTRUCTION OF ORTHOGONAL BASIS FUNCTIONS

Let the function $q_n(x)$ defined as

$$q_n(x) = \sum_{r=0}^n C_r^{(n)} x^r \tag{2}$$

where $C_r^{(n)}$'s are the orthogonal coefficients and $q_n(x)$ satisfies the inner product

$$\langle q_m(x), q_n(x) \rangle = \int_a^b w(x) q_m(x) q_n(x) dx = 0, \quad m \neq n, [-1, 1] \tag{3}$$

For the purpose of constructing the basis function, we use additional property that

$$q_n(1) = 1 \tag{4}$$

For $n = 0$ in (2),

$$q_0(x) = C_0^{(0)}$$

From (4),

$$q_0(1) = C_0^{(0)} = 1$$

Hence,

$$q_0(x) = 1$$

For $n = 1$ in (2),

$$q_1(x) = C_0^{(1)} + C_1^{(1)} x \tag{5}$$

By definition (4), (5) gives

$$C_0^{(1)} + C_1^{(1)} = 1 \tag{6}$$

and

$$\langle q_0, q_1 \rangle = \int_0^1 \left(1 + \frac{x}{2}\right) q_0(x) q_1(x) dx \tag{7}$$

which implies

$$\frac{5}{4} C_0^{(1)} + \frac{2}{3} C_1^{(1)} = 0 \tag{8}$$

Solving (6) and (10) and substituting the outcomes into (5), we have

$$q_1(x) = \frac{1}{7}(15x - 8) \tag{9}$$

When $n = 2$ in (2),

$$q_2(x) = C_0^{(2)} + C_1^{(2)} x + C_2^{(2)} x^2$$

By definition (4), (10) gives

$$C_0^{(2)} + C_1^{(2)} + C_2^{(2)} = 1 \tag{10}$$

and

$$\langle q_0, q_2 \rangle = \int_0^1 \left(1 + \frac{x}{2}\right) q_0(x) q_2(x) dx = 0 \tag{12}$$

which implies

$$\frac{5}{4} C_0^{(2)} + \frac{2}{3} C_1^{(2)} + \frac{11}{24} C_2^{(2)} = 0 \tag{13}$$

Also

$$\langle q_1, q_2 \rangle = \int_0^1 \left(1 + \frac{x}{2}\right) q_1(x) q_2(x) dx \tag{14}$$

which gives

$$\frac{37}{168} C_1^{(2)} + \frac{19}{84} C_2^{(2)} = 0 \tag{15}$$

Solving (11), (13), (15) and substituting the resulting values into (10), we have

$$q_2(x) = \frac{1}{57}(370x^2 - 380x + 67) \tag{16}$$

When $n = 3$ in (2),

$$q_3(x) = C_0^{(3)} + C_1^{(3)} x + C_2^{(3)} x^2 + C_3^{(3)} x^3 \tag{17}$$

By definition (4), (17) gives

$$C_0^{(3)} + C_1^{(3)} + C_2^{(3)} + C_3^{(3)} = 1 \tag{18}$$

$$\langle q_0, q_3 \rangle = \int_0^1 \left(1 + \frac{x}{2}\right) q_0(x) q_3(x) dx = 0 \tag{19}$$

which implies

$$\frac{5}{4} C_0^{(3)} + \frac{2}{3} C_1^{(3)} + \frac{11}{24} C_2^{(3)} + \frac{7}{20} C_3^{(3)} = 0 \tag{20}$$

$$\langle q_1, q_3 \rangle = \int_0^1 \left(1 + \frac{x}{2}\right) q_1(x) q_3(x) dx = 0 \tag{21}$$

This leads to

$$\frac{37}{168} C_1^{(3)} + \frac{19}{84} C_2^{(3)} + \frac{29}{140} C_3^{(3)} = 0 \tag{22}$$

$$\langle q_2, q_3 \rangle = \int_0^1 \left(1 + \frac{x}{2}\right) q_2(x) q_3(x) dx = 0 \tag{23}$$

Solving (18),(20) and (22) and substituting the resulting values into (17), we obtain

$$q_3(x) = \frac{1}{491}(10675x^3 - 16290x^2 + 6690x - 584) \tag{24}$$

In the same vein, $q_n(x), n \geq 4$ are developed. The next three polynomials which are used in this work are listed hereunder.

2.1 FORMULATION OF THE NUMERICAL INTEGRATION

$$q_4(x) = \frac{1}{4361}(332766x^4 - 674072x^3 + 440874x^2 - 100428x + 5221)$$

$$q_5(x) = \frac{1}{7899}(2173710x^5 - 5489736x^4 + 4942812x^3 - 1884904x^2 + 275513x - 9496)$$

$$q_6(x) = \frac{1}{72509}(73254324x^6 - 221626152x^5 + 254436138x^4 - 137426374x^3 + 34913052x^2 - 3565896x + 87419)$$

we review here the work of Adam-Moulton on the derivation of three-step implicit method whose discrete scheme is $y_{n+3} = y_{n+2} + \frac{h}{24}(f_n - 5f_{n+1} + 19f_{n+2} + 9f_{n+3})$ so as to investigate the applicability of the derived orthogonal polynomials.

For this purpose, we shall seek an approximation of the form $y(x) = \sum_{r=0}^{s+k-1} a_r q_r(x)$ (25)

Where $q_r(x)$ is the orthogonal polynomials derived.

Equation (25) is collocated and interpolated at $x = x_{n+i}, i = 0(1)3$ and $x = x_{n+2}$ respectively to obtain a system of equations which are solved and the resulting values of a_r are substituted back into (25) to have a continuous scheme. Evaluating the continuous scheme at the grid point $x = x_{n+3}$ yields the Adams-Moulton explicit three-step method.

A set of polynomials shall now be employed to formulate a continuous scheme through which numerical solutions of initial value problems in ordinary differential equations are obtained.

Our objective in this section is to derive a two-step continuous hybrid linear multistep method in the sub-interval $[x_n, x_{n+p}]$ of $[a, b]$ where $x = \frac{2x - 2x_n - ph}{ph}$ and p varies as the method to be derived. For this case

$$p = 2.$$

The procedure involves interpolating (25) at $x = x_{k+i}, i = 0, \frac{1}{2}, 1$ and collocating the third derivative of (25) at $x = x_{k+i}, i = 0, \frac{1}{2}, 1, \frac{3}{2}$ and 2.

The $a_r (0 \leq r \leq 7)$, from the resulting system of equations are obtained and substituted into (25) to have the continuous equation

$$y(x) = \alpha_0(x)y_k + \alpha_{\frac{1}{2}}(x)y_{k+\frac{1}{2}} + \alpha_1(x)y_{k+1} + h^3 \left(\sum_{j=0}^2 \beta_j(x)f_{k+j} + \beta_{\frac{1}{2}}(x)f_{k+\frac{1}{2}} + \beta_{\frac{3}{2}}(x)f_{k+\frac{3}{2}} \right) \quad (26)$$

Evaluating equation (26) at $x = x_{k+\frac{3}{2}}$ and $x = x_{k+2}$ yield the following main methods as

$$y_{k+\frac{3}{2}} = y_k - 3y_{k+\frac{1}{2}} + 3y_{k+1} + \frac{h^3}{1920} \left(f_k + 116f_{k+\frac{1}{2}} + 126f_{k+1} - 4f_{k+\frac{3}{2}} + f_{k+2} \right) \quad (27)$$

$$y_{k+2} = 3y_k - 8y_{k+\frac{1}{2}} + 6y_{k+1} + \frac{h^3}{480} \left(f_k + 86f_{k+\frac{1}{2}} + 126f_{k+1} + 26f_{k+\frac{3}{2}} + f_{k+2} \right) \quad (28)$$

The general block formula proposed in [10] in the normalized form given as

$$A^{(0)}Y_m = ey_m + h^{\mu-\tau}df(y_m) + h^{\mu-\tau}bF(y_m) \quad (29)$$

shall be adopted in order to develop the block formula from the continuous scheme.

Evaluating the first and second derivatives of (26) at $x = x_{k+i}, i = 0, \frac{1}{2}, 1, \frac{3}{2}$ and 2 and substituting the resulting equations and the main methods (27), (28) into (29) and solving simultaneously gives a block formula represented as

$$\begin{aligned} y_{k+\frac{1}{2}} &= y_k + \frac{1}{2}hy'_k + \frac{1}{8}h^2y''_k + \frac{113}{8960}h^3f_k - \frac{103}{13440}h^3f_{k+1} - \frac{47}{80640}h^3f_{k+2} + \frac{107}{8064}h^3f_{k+\frac{1}{2}} + \frac{43}{13440}h^3f_{k+\frac{3}{2}} \\ y_{k+1} &= y_k + hy'_k + \frac{1}{2}h^2y''_k + \frac{331}{5040}h^3f_k - \frac{1}{21}h^3f_{k+1} - \frac{19}{5040}h^3f_{k+2} + \frac{83}{630}h^3f_{k+\frac{1}{2}} + \frac{13}{630}h^3f_{k+\frac{3}{2}} \\ y_{k+\frac{3}{2}} &= y_k + \frac{3}{2}hy'_k + \frac{9}{8}h^2y''_k + \frac{1431}{8960}h^3f_k - \frac{243}{4480}h^3f_{k+1} - \frac{81}{8960}h^3f_{k+2} + \frac{1863}{4480}h^3f_{k+\frac{1}{2}} + \frac{45}{896}h^3f_{k+\frac{3}{2}} \\ y_{k+2} &= y_k + 2hy'_k + 2h^2y''_k + \frac{31}{105}h^3f_k + \frac{4}{105}h^3f_{k+1} - \frac{1}{63}h^3f_{k+2} + \frac{272}{315}h^3f_{k+\frac{1}{2}} + \frac{16}{105}h^3f_{k+\frac{3}{2}} \\ y'_{k+\frac{1}{2}} &= hy'_k + h^2y''_k + \frac{53}{360}h^3f_k - \frac{1}{12}h^3f_{k+1} - \frac{1}{120}h^3f_{k+2} + \frac{2}{5}h^3f_{k+\frac{1}{2}} + \frac{2}{45}h^3f_{k+\frac{3}{2}} \\ y'_{k+1} &= hy'_k + \frac{3}{2}h^2y''_k + \frac{147}{640}h^3f_k + \frac{27}{320}h^3f_{k+1} - \frac{9}{640}h^3f_{k+2} + \frac{117}{160}h^3f_{k+\frac{1}{2}} + \frac{3}{32}h^3f_{k+\frac{3}{2}} \\ y'_{k+\frac{3}{2}} &= hy'_k + \frac{1}{2}h^2y''_k + \frac{367}{5760}h^3f_k - \frac{47}{960}h^3f_{k+1} - \frac{7}{1960}h^3f_{k+2} + \frac{3}{32}h^3f_{k+\frac{1}{2}} + \frac{29}{1440}h^3f_{k+\frac{3}{2}} \\ y'_{k+2} &= hy'_k + 2h^2y''_k + \frac{14}{45}h^3f_k + \frac{4}{15}h^3f_{k+1} + \frac{16}{15}h^3f_{k+\frac{1}{2}} + \frac{16}{45}h^3f_{k+\frac{3}{2}} \\ y''_{k+\frac{1}{2}} &= h^2y''_k + \frac{29}{180}h^3f_k + \frac{2}{15}h^3f_{k+1} - \frac{1}{180}h^3f_{k+2} + \frac{31}{45}h^3f_{k+\frac{1}{2}} + \frac{1}{45}h^3f_{k+\frac{3}{2}} \\ y''_{k+1} &= h^2y''_k + \frac{27}{160}h^3f_k + \frac{9}{20}h^3f_{k+1} - \frac{3}{160}h^3f_{k+2} + \frac{51}{80}h^3f_{k+\frac{1}{2}} + \frac{21}{80}h^3f_{k+\frac{3}{2}} \\ y''_{k+\frac{3}{2}} &= h^2y''_k + \frac{251}{1440}h^3f_k - \frac{11}{60}h^3f_{k+1} - \frac{19}{1440}h^3f_{k+2} + \frac{323}{720}h^3f_{k+\frac{1}{2}} + \frac{53}{720}h^3f_{k+\frac{3}{2}} \\ y''_{k+2} &= h^2y''_k + \frac{7}{45}h^3f_k + \frac{4}{15}h^3f_{k+1} + \frac{7}{45}h^3f_{k+2} + \frac{32}{45}h^3f_{k+\frac{1}{2}} + \frac{32}{45}h^3f_{k+\frac{3}{2}} \end{aligned} \quad (30)$$

3.0 BASIC PROPERTIES OF THE METHOD

Definition 3.1.1 Order and Error Constant

The linear operator L of the block (29) is defined as

$$L\{y(x); h\} = Y_m - ey_m + h^{\mu-\tau}df(y_m) + h^{\mu-\tau}bF(y_m) \quad (31)$$

Using Taylor expansion to expansion to expand $y(x_n + ih)$ and $f(x_n + jh)$, (31) becomes

$$L\{y(x); h\} = C_0y(x) + C_1hy'(x) + C_2h^2y''(x) + \dots + C_ph^py^{(p)}(x) \quad (32)$$

$$x''' + Ax'' + Bx' - f(x(t)) = 0$$

The block (29) and associated linear operator are said to have order p if

$$C_0 = C_1 = C_2 = \dots + C_p = C_{p+1} = C_{p+2} \neq 0$$

The term $C_{p+2} \neq 0$ is called the error constant and the local truncation error is given as $t_{n+k} = C_{p+2}h^{(p+2)}y^{(p+2)}(x_n) + 0h^{(p+3)}$

Thus, equation (27) and (28) each are of order 5 and error constant

$$\frac{243}{11468800} \text{ and } \frac{139}{10321920} \text{ respectively. The formulae in the block (30) are all of}$$

order 5 with error constants

$$C_{p+3} = \left[\frac{1}{11520}, \frac{23}{1814400}, \frac{829}{1124800}, \frac{61}{3444525}, \frac{13}{453600}, -\frac{11}{14175}, \frac{211}{5248800}, \frac{7}{328050}, \frac{1}{21600}, -\frac{1}{450} \right]$$

Respectively.

Definition 3.1.2 Zero-Stability

The block (29) is said to be zero stable if the roots $Z_s = 1, 2 \dots N$ of the characteristic polynomial $\rho(z) = \det(zA - E)$ satisfies $|z| \leq 1$ and the root $|z| = 1$ has multiplicity not exceeding the order of the differential equation. Also, as $h^\mu \rightarrow 0$, $\rho(z) = z^{r-\mu}(\tau - 1)^\mu$, where μ is the order of the differential equation, $r = \dim(A^{(0)})$.

Thus, the proposed method has been investigated to be zero stable.

Definition 3.1.3 Consistency

A numerical method is consistent if the order, $p \geq 1$

Since the order of the derived method is, $p \geq 1$, the method is said to be consistent.

Definition 3.1.4 Convergence

The necessary and sufficient condition for a numerical method to be convergent is for it to be zero-stable and consistent. According to the definition, the method derived is convergent

4.0 Numerical Applications

We consider here the application of the derived schemes to three test problems for the efficiency and accuracy of the method implemented as block method

Problem 4.1.1: (A Non-linear problem)

$$y^2y''' = 1, \quad y(0) = 1 \quad y'(0) = 1, \quad y''(0) = 1, \quad h = 0.1$$

Source: [11]

The above problem was derived by Tanner to investigate the motion of the contact line for a thin oil drop spreading on a horizontal surface.

Problem 4.1.2 Non-linear Blasius Equation (Application Problem)

$$2y''' + yy'' = 0$$

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1$$

The exact solution does not exist.

Source: [2]

Problem 4.1.3 Non-linear Genesio Equation (Application Problem)

Here we consider the nonlinear chaotic system from Genesio and Tesi (1992)

with

$$f(x(t)) = -Cx(t) + x^2(t)$$

that is subject to the following initial conditions:

$$x(0) = 0.2, \quad x'(0) = -0.3, \quad x''(0) = 0.1, \quad t \in [0, b],$$

where $A = 1.2$, $B = 2.29$ and $C = 6$ are positive constants that satisfied $AB < C$ for the existence of the solution.

Source: [15]

TABLE 4.1.1: COMPARING THE SOLUTION OF THE APPROXIMATE AND THE EXISTING METHOD FOR PROBLEM 4.1.1

X	Exact Solution	Result of New Method	Error in New Method Order P = 5	Error in Tanner (1979) Order P = 4
0.2	1.22121001 3377463526 20	1.22121000 4011691528 60	9.36577199 $\times 10^{-09}$	2.40500000 $\times 10^{-05}$
0.4	1.48883473 2966371756 50	1.48883477 6509767172 00	4.35433954 $\times 10^{-08}$	7.71670000 $\times 10^{-05}$
0.6	1.80736134 9197207648 40	1.80736138 8157622586 00	3.89604149 $\times 10^{-08}$	7.94945000 $\times 10^{-06}$
0.8	2.17981922 6249380859 50	2.17981921 4596906729 30	1.16524741 $\times 10^{-08}$	4.34949000 $\times 10^{-03}$
1.0	2.60827491 8352179410 00	2.60827483 4743114714 80	8.36090470 $\times 10^{-08}$	1.83199620 $\times 10^{-02}$

TABLE 4.1.2: COMPARING THE SOLUTION OF THE APPROXIMATE AND THE EXISTING METHOD FOR PROBLEM 4.1.2

X	Exact Solution	Result of New Method	Error in New Method Order P = 5	Error in Adesanya [2] Order P = 6
0.1	0.00499995 5187456010 00	0.00499995 8330580560 99	$3.14312455 \times 10^{-09}$	$4.27300000 \times 10^{-08}$
0.2	0.01999865 9080238100 00	0.01999866 6823456901 30	$7.74321880 \times 10^{-09}$	$1.20759000 \times 10^{-06}$
0.3	0.04498987 4102594700 00	0.04498987 9427407235 00	$5.32481254 \times 10^{-09}$	$8.60719000 \times 10^{-06}$
0.4	0.07995737 7351676100 00	0.07995737 7883886230 47	$5.32210130 \times 10^{-10}$	$3.40900400 \times 10^{-05}$
0.5	0.12487004 7646537000 00	0.12487005 7334906231 82	$9.68836923 \times 10^{-09}$	$9.74068000 \times 10^{-05}$
0.6	0.17967712 6361217000 00	0.17967714 1009941594 50	$1.46487246 \times 10^{-08}$	$2.25711000 \times 10^{-04}$
0.7	0.24430361 2900385000 00	0.24430361 6586205861 18	$3.68582086 \times 10^{-09}$	$4.51454700 \times 10^{-04}$
0.8	0.31864597 9464674000 00	0.31864600 8680488155 89	$2.92158142 \times 10^{-08}$	$8.08472900 \times 10^{-04}$
0.9	0.40256860 6213134000 00	0.40256861 9610775502 41	$1.33976415 \times 10^{-08}$	$1.32622070 \times 10^{-03}$
1.0	0.49590033 7629337000 00	0.49590038 1439983734 66	$4.38106467 \times 10^{-08}$	$2.02205460 \times 10^{-03}$

TABLE 4.1.3: ABSOLUTE ERRORS COMPARING THE EXACT AND NUMERICAL SOLUTION OF HTEPM FOR PROBLEM 4.1.3

X	Exact Solution	Result of New Method	Error in New Method Order P = 5
0.1	0.170440346269364	0.17044034739217347603	$1.12280948 \times 10^{-09}$
0.2	0.141582173138664	0.14158217447659620875	$1.33793221 \times 10^{-09}$
0.3	0.113282963581607	0.11328296817008674018	$4.58847974 \times 10^{-09}$
0.4	0.085554524922736	0.08555454145299195369	$1.65302560 \times 10^{-08}$
0.5	0.058543682864592	0.05854370999334741200	$2.71287546 \times 10^{-08}$
0.6	0.032510877478247	0.03251091543736197638	$3.79591149 \times 10^{-08}$
0.7	0.007806854082744	0.00780690667931403256	$5.25965700 \times 10^{-08}$
0.8	-	-	7.23143847
0.9	0.151523368042584	0.1515226448987367326	$9.66923835 \times 10^{-08}$
1.0	0.359116451185857	0.3591154842620219391	$1.23420860 \times 10^{-07}$

5.0 DISCUSSION OF RESULTS

Problems 4.1.1 is a non-linear problem derived by Tanner to investigate the motion of the contact line for a thin oil drop spreading on a horizontal surface. Problem 4.1.2 considered Blassius equation in thermodynamics. The non-linear Genesio equation of problem 4.1.3 is a non-linear chaotic system from [15]. The results were displayed in Tables 4.1.1, 4.1.2 and 4.1.3 respectively. The absolute errors obtained from tables 4.1.1 and 4.1.2 revealed that on comparison with the exact solution, the low errors resulted demonstrate their effectiveness and accuracy as the schemes performed favorably well. The exact solution, however, for problems 4.1.1, 4.1.2, and 4.1.3 were not available. Hence, they were generated directly using Maple software environment.

6.0 CONCLUSION

The construction of a new class of continuous implicit two-step hybrid scheme capable of solving Initial Value problems of third order ODEs has been the central concern in this work. The Orthogonal Polynomials valid in the interval $[-1,1]$ with respect to weight function $w(x) = 1 + \frac{x}{2}$ have been chosen as basis functions to develop the schemes using interpolation and

collocation techniques with the incorporation of equally spaced off-step points in order to approximate the solutions of IVPs. The scheme is capable of handling non-linear application problems. Tables 1, 2 and 3 displays the accuracy of the numerical results of the HTEPM with the exact solution and existing methods. The desirability and superiority of the method have been established by the numerical results.

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