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RESEARCH ARTICLE

TRIGONOMETRICALLY FITTED BLOCK METHOD FOR SOLVING SECOND-ORDER INITIAL VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

This study focuses on developing an implicit continuous four-step method for the numerical solution of initial value problems (IVPs) stemming from ordinary differential equations(ODEs) of second-order. A combination of polynomial and trigonometric basis functions serves as the backbone of this method. The derivation process employs interpolation and collocation techniques, facilitating a robust framework for solving IVPs. Through rigorous analysis employing pertinent theorems, the scheme's consistency, convergence and stability were meticulously scrutinized. The findings of this investigation assert that the developed method exhibits consistency, zero-stability and consequently convergence, underlining its robustness and reliability in practical applications in comparison to existing literature in terms of accuracy, the proposed method outperforms the existing authors when the error are compared. In essence, this work represents a significant advancement in numerical methods for solving IVPs of second-order ODEs. The study provides a strong basis for future research in this area by utilizing a unique combination of polynomial with the sum of sine and cosine as the basis function and putting the approach through theoretical scrutiny. This opens up intriguing directions for further investigation and improvement.

KEYWORDS

Collocation, Interpolation, Convergence, Stability

Introduction

This paper presents a numerical method for solving initial value problems (IVP) for general second-order ODEs of the form.

$$y'' = f(x, y, y'), \quad y(a) = \eta_0, \; y'(a) = \eta_1 \quad x\varepsilon[a, b] \tag{1}$$

Second-order linear differential equations find applications in various domains such as physics and engineering, offering a framework to model system of equations. In the realm of numerical integration, particularly for oscillatory problems, the exploration began in 1961 with Gautsci introducing the Adams and St¨ormer order using trigonometric polynomials.

Higher order ODEs in ordinary differential equations were first reduced to a system of first-order ODEs, allowing for the application of any suitable method for solving first-order ODEs.Butcher

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(2018),Fatunla (1991) and Awoyemi (2001) have studied the reduction procedure in great detail.

Building upon Gautsci groundwork, Psihoyios and Simos (2003, 2005) introduced trigonometrically fitted schemes for solving oscillatory problems. These schemes, operating in predictor-corrector mode, employed the Adams-Bashforth method as the predictor and Adams-Moulton as the corrector. However, their implementation posed challenges, requiring substantial human effort and often resulting in diminished accuracy because the predictor and corrector are usually of different orders.

Addressing these limitations, Jator et. al. (2012) delved into multistep collocation methods to devise trigonometrically fitted approaches based on trigonometric polynomials, introducing Numerov type block methods. Ngwane and Jator (2020) expanded on this by developing a block hybrid scheme specifically tailored for integrating oscillatory problems.

To overcome the drawbacks of the predictor-corrector method, researchers adopted the block method, which can begin computations without requiring previously computed values or additional initialization steps. Noteworthy contributors in this domain include Abolarin et. al.(2020), Omar and Kuboye (2018),Olanegan et. al. (2018) and Awoyemi et. al. (2015) among others.

In this study, an approach involving a second derivative trigonometrically fitted method is presented, utilizing the multistep collocation technique. The approximated interpolating function is constructed as a linear combination of polynomials and trigonometric terms. Specifically, our proposed method is designed to precisely integrate initial value problems (IVPs) stated as combinations of the set $\{1, x, x^2, x^3, x^4, \sin(\varpi x), \cos(\varpi x)\}$. This basis function is chosen for its analytical simplicity and its ability to offer an enhanced framework for addressing initial value problems characterized by oscillatory solutions. The combination of these two functions will help to produce more accurate and stable results.

Methodology

In developing this method, the combination of polynomials with sine and cosine functions of the form

$$y(x) = \sum_{n=0}^{k} a_n x^n + a_5 \sin(\varpi x) + a_6 \cos(\varpi x) \tag{2}$$

is considered as an approximate solution to equation (1). Where the coefficients *a s* are arbitrary real constant that must be equally determined, the frequency of the trigonometric function ($) will be employed to enhance the method's accuracy.

The second derivative of equation (2) is obtained as:

$$y'' = f(x, y, y'), \quad y(a) = \eta_0, \; y'(a) = \eta_1 \quad x\varepsilon[a, b] \tag{3}$$

Collocating equation (3) at $x = x_{n+j}$, $j = 0(1)k$ and interpolating equation (2) at $x = x_{n+j}$, $j = 0,1$ gives a seven non-singular equations which can be written as a system in matrix form in equation (4)

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & \sin\varpi x_n & \cos\varpi x_n \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & \sin\varpi x_{n+1} & \cos\varpi x_{n+1} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & -\varpi^2\sin\varpi x_n & -\varpi^2\cos\varpi x_n \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & -\varpi^2\sin\varpi x_{n+1} & -\varpi^2\cos\varpi x_{n+1} \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & -\varpi^2\sin\varpi x_{n+2} & -\varpi^2\cos\varpi x_{n+2} \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & -\varpi^2\sin\varpi x_{n+3} & -\varpi^2\cos\varpi x_{n+3} \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & -\varpi^2\sin\varpi x_{n+4} & -\varpi^2\cos\varpi x_{n+4} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+1} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} \tag{4}$$

Then solving equation (4) using Gaussian elimination method with the help of Computer aided software gives values for the unknown parameters $a_n(s)$ which gives our continuous implicit method when substituted into equation (2)

$$y(x) = \sum_{j=0}^{k} \alpha_j(x) y_{n+j} + h^2 \left[\sum_{j=0}^{k} \beta_j(x) f_{n+j} \right] \tag{5}$$

where $\alpha_j(x)$and $\beta_j(x)$ are continuous coefficients, $y_{n+j} = y(x_n + jh)$ is the numerical approximation of the analytical solution at $x_{n+j}$and $f_{n+j} = f(x_{n+j}, y_{n+j}, y'_{n+j})$.

Using $t = \frac{x - x_n}{h}$, $\frac{dt}{dx} = \frac{1}{h}$, equation (5) is evaluated at the non-interpolating points x_{n+j};$j = 2(1)k$ while the

first derivative of Equation (5) is evaluated at all points, their coefficients and the corresponding Taylor series conversion up to $O(u^8)$ by letting u = $h are given as follows

$$\alpha_1(t) = t$$

$$\begin{aligned} \beta_0(t) = \frac{1}{24}\Bigg(& \frac{24\sin(h\varpi) - 72\sin(2h\varpi) + 72\sin(3h\varpi) - 24\sin(4h\varpi) - 96t\sin(h\varpi) + 144t\sin(2h\varpi) - 96t\sin(3h\varpi)}{\varpi^2(14\sin(h\varpi) - 14\sin(2h\varpi) + 6\sin(3h\varpi) - \sin(4h\varpi))} \\ & + \frac{24t\sin(4h\varpi) - 111h^2t\varpi^2\sin(h\varpi) + 96h^2t\varpi^2\sin(2h\varpi) - 27h^2t\varpi^2\sin(3h\varpi) + 156h^2t^2\varpi^2\sin(h\varpi)}{\varpi^2(14\sin(h\varpi) - 14\sin(2h\varpi) + 6\sin(3h\varpi) - \sin(4h\varpi))} \\ & + \frac{-50h^2t^3\varpi^2\sin(h\varpi) - 132h^2t^2\varpi^2\sin(2h\varpi) + 5h^2t4\varpi^2\sin(h\varpi) + 36h^2t^2\varpi^2\sin(3h\varpi) + 40h^2t^3\varpi^2\sin(2h\varpi)}{\varpi^2(14\sin(h\varpi) - 14\sin(2h\varpi) + 6\sin(3h\varpi) - \sin(4h\varpi))} \\ & + \frac{-10h^2t^3\varpi^2\sin(3h\varpi) - 4h^2t^4\varpi^2\sin(2h\varpi) + h^2t^4\varpi^2\sin(3h\varpi)}{\varpi^2(14\sin(h\varpi) - 14\sin(2h\varpi) + 6\sin(3h\varpi) - \sin(4h\varpi))}\Bigg) \end{aligned}$$

$$\begin{aligned} \beta_1(t) = \frac{-1}{24}\Bigg(& \frac{-144\sin(2h\varpi) + 192\sin(3h\varpi) - 72\sin(4h\varpi) - 168t\sin(h\varpi) + 336t\sin(2h\varpi) - 264t\sin(3h\varpi)}{\varpi^2(14\sin(h\varpi) - 14\sin(2h\varpi) + 6\sin(3h\varpi) - \sin(4h\varpi))} \\ & + \frac{+72t\sin(4h\varpi) + 34h^2t\varpi^2\sin(h\varpi) - 74h^2t\varpi^2\sin(2h\varpi) + 74h^2t\varpi^2\sin(3hw) - 27h^2t\varpi^2\sin(4h\varpi) - 40h^2t^3\varpi^2\sin(h}{\varpi^2(14\sin(h\varpi) - 14\sin(2h\varpi) + 6\sin(3h\varpi) - \sin(4h\varpi))} \\ & + \frac{72h^2t^2\varpi^2\sin(2h\varpi) + 6h^2t^4\varpi^2\sin(h\varpi) - 96h^2t^2\varpi^2\sin(3h\varpi) + 4h^2t^3\varpi^2\sin(2h\varpi) + 36h^2t^2\varpi^2\sin(4h\varpi)}{\varpi^2(14\sin(h\varpi) - 14\sin(2h\varpi) + 6\sin(3h\varpi) - \sin(4h\varpi))} \\ & + \frac{24h^2t^3\varpi^2\sin(3h\varpi) - 2h^2t^4\varpi^2\sin(2h\varpi) - 10h^2t^3\varpi^2\sin(4h\varpi) - 2h^2t^4\varpi^2\sin(3h\varpi) + h^2t^4\varpi^2\sin(4h\varpi)}{\varpi^2(14\sin(h\varpi) - 14\sin(2h\varpi) + 6\sin(3h\varpi) - \sin(4h\varpi))}\Bigg) \end{aligned}$$

$$\begin{aligned} \beta_2(t) = \frac{1}{12}\Bigg(& \frac{-72\sin(h\varpi) + 72\sin(3h\varpi) - 36\sin(4h\varpi) + 36t\sin(h\varpi) + 72t\sin(2h\varpi) - 108t\sin(3h\varpi) + 36t\sin(4h\varpi)}{\varpi^2(14\sin(h\varpi) - 14\sin(2h\varpi) + 6\sin(3h\varpi) - \sin(4h\varpi))} \\ & + \frac{29h^2t\varpi^2\sin(h\varpi) + 14h^2t\varpi^2\sin(2h\varpi) + 15h^2t\varpi^2\sin(3h\varpi) - 11h^2t\varpi^2\sin(4h\varpi) + 36h^2t^2\varpi^2\sin(h\varpi)}{\varpi^2(14\sin(h\varpi) - 14\sin(2h\varpi) + 6\sin(3h\varpi) - \sin(4h\varpi))} \\ & + \frac{8h^2t^3\varpi^2\sin(h\varpi) + h^2t^4\varpi^2\sin(h\varpi) - 36h^2t^2\varpi^2\sin(3h\varpi) - 16h^2t^3\varpi^2\sin(2h\varpi) + 18h^2t^2\varpi^2\sin(4h\varpi)}{\varpi^2(14\sin(h\varpi) - 14\sin(2h\varpi) + 6\sin(3h\varpi) - \sin(4h\varpi))} \\ & + \frac{24h^2t^3\varpi^2\sin(3h\varpi) + 2h^2t^4\varpi^2\sin(2h\varpi) - 8h^2t^3\varpi^2\sin(4h\varpi) - 3h^2t^4\varpi^2\sin(3h\varpi) + h^2t^4\varpi^2\sin(4h\varpi)}{\varpi^2(14\sin(h\varpi) - 14\sin(2h\varpi) + 6\sin(3h\varpi) - \sin(4h\varpi))}\Bigg) \end{aligned}$$

$$\begin{aligned} \beta_3(t) = \frac{-1}{24}\Bigg(& \frac{-192\sin(h\varpi) + 144\sin(2h\varpi) - 24\sin(4h\varpi) + 264t\sin(h\varpi) - 144t\sin(2h\varpi) - 24t\sin(3h\varpi)}{\varpi^2(14\sin(h\varpi) - 14\sin(2h\varpi) + 6\sin(3h\varpi) - \sin(4h\varpi))} \\ & + \frac{24t\sin(4h\varpi) - 46h^2t\varpi^2\sin(h\varpi) + 46h^2t\varpi^2\sin(2h\varpi) - 6h^2t\varpi^2\sin(3h\varpi) - 7h^2t\varpi^2\sin(4h\varpi) + 96h^2t^2\varpi^2\sin(h\varpi)}{\varpi^2(14\sin(h\varpi) - 14\sin(2h\varpi) + 6\sin(3h\varpi) - \sin(4h\varpi))} \\ & + \frac{-56h^2t^3\varpi^2\sin(h\varpi) - 72h^2t^2\varpi^2\sin(2h\varpi) + 6h^2t^4\varpi^2\sin(h\varpi) + 28h^2t^3\varpi^2\sin(2h\varpi) + 12h^2t^2\varpi)^2\sin(4h\varpi)}{\varpi^2(14\sin(h\varpi) - 14\sin(2h\varpi) + 6\sin(3h\varpi) - \sin(4h\varpi))} \\ & + \frac{8h^2t^3\varpi^2\sin(3h\varpi) - 2h^2t^4\varpi^2\sin(2h\varpi) - 6h^2t^3\varpi^2\sin(4h\varpi) - 2h^2t^4\varpi^2\sin(3h\varpi) + h^2t^4\varpi^2\sin(4h\varpi)}{\varpi^2(14\sin(h\varpi) - 14\sin(2h\varpi) + 6\sin(3h\varpi) - \sin(4h\varpi))}\Bigg) \end{aligned}$$

$$\begin{aligned} \beta_4(t) = \frac{1}{24}\Bigg(& \frac{-72\sin(h\varpi) + 72\sin(2h\varpi) - 24\sin(3h\varpi) + 120t\sin(h\varpi) - 96t\sin(2h\varpi) + 24t\sin(3h\varpi)}{\varpi^2(14\sin(h\varpi) - 14\sin(2h\varpi) + 6\sin(3h\varpi) - \sin(4h\varpi))} \\ & + \frac{-11h^2t\varpi^2\sin(h\varpi) + 16h^2t\varpi^2\sin(2h\varpi) - 7h^2t\varpi^2\sin(3h\varpi) + 36h^2t^2\varpi^2\sin(h\varpi) - 30h^2t^3\varpi^2\sin(h\varpi)}{\varpi^2(14\sin(h\varpi) - 14\sin(2h\varpi) + 6\sin(3h\varpi) - \sin(4h\varpi))} \\ & + \frac{-36h^2t^2\varpi^2\sin(2h\varpi) + 5h^2t^4\varpi^2\sin(h\varpi) + 12h^2t^2\varpi^2\sin(3h\varpi) + 24h^2t^3\varpi^2\sin(2h\varpi) - 6h^2t^3\varpi^2\sin(3h\varpi)}{\varpi^2(14\sin(h\varpi) - 14\sin(2h\varpi) + 6\sin(3h\varpi) - \sin(4h\varpi))} \\ & + \frac{-4h^2t^4\varpi^2\sin(2h\varpi) + h^2t^4\varpi^2\sin(3h\varpi)}{\varpi^2(14\sin(h\varpi) - 14\sin(2h\varpi) + 6\sin(3h\varpi) - \sin(4h\varpi))}\Bigg) \end{aligned}$$

$\alpha_0(t) = -(t - 1)$

$$\alpha_0(t) = -t \tag{6}$$

The result is expressed in series form as follows when evaluated at $t = 2$:

$$\left.\begin{aligned}
y_{n+2}-2y_{n+1}+y_n &= h^2\left[\beta_0(x)f_n+\beta_1(x)f_{n+1}+\beta_2(x)f_{n+2}+\beta_3(x)f_{n+3}+\beta_4(x)f_{n+4}\right],\\
\beta_0 &= \frac{19}{240}h^2+\frac{221}{60480}h^2u^2\ \frac{233}{1814400}h^2u^4+\frac{199}{53222400}h^2u^6+O(u^8),\\
\beta_1 &= -\frac{17}{20}h^2+\frac{79}{7560}h^2u^2+\frac{79}{226800}h^2u^4+\frac{61}{6652800}h^2u^6+O(u^8),\\
\beta_2 &= \frac{7}{120}h^2+\frac{19}{2016}h^2u^2+\frac{83}{302400}h^2u^4+\frac{1}{197120}h^2u^6+O(u^8),\\
\beta_3 &= \frac{1}{60}h^2-\frac{2}{945}h^2u^2-\frac{1}{56700}h^2u^4+\frac{1}{415800}h^2u^6+O(u^8),\\
\beta_4 &= -\frac{1}{240}h^2-\frac{31}{60480}h^2u^2-\frac{67}{1814400}h^2u^4-\frac{109}{53222400}h^2u^6+O(u^8)
\end{aligned}\right\} \tag{7}$$

The result is expressed in series form as follows when evaluated at $t = 3$:

$$\left.\begin{aligned}
y_{n+3}-3y_{n+1}+2y_n &= h^2\left[\beta_0(x)f_n+\beta_1(x)f_{n+1}+\beta_2(x)f_{n+2}+\beta_3(x)f_{n+3}+\beta_4(x)f_{n+4}\right],\\
\beta_0 &= \frac{37}{240}h^2+\frac{137}{20160}h^2u^2+\frac{19}{86400}h^2u^4\ \frac{289}{53222400}h^2u^6+O(u^8),\\
\beta_1 &= -\frac{9}{5}h^2+\frac{19}{1008}h^2u^2+\frac{83}{151200}h^2u^4+\frac{1}{98560}h^2u^6+O(u^8),\\
\beta_2 &= \frac{37}{40}h^2+\frac{53}{3360}h^2u^2+\frac{11}{33600}h^2u^4-\frac{19}{8870400}h^2u^6+O(u^8),\\
\beta_3 &= \frac{2}{15}h^2-\frac{11}{540}h^2u^2+\frac{17}{151200}h^2u^4+\frac{173}{13305600}h^2u^6+O(u^8),\\
\beta_4 &= \frac{1}{80}h^2+\frac{31}{20160}h^2u^2\ \frac{67}{604800}h^2u^4+\frac{109}{17740800}h^2u^6+O(u^8)
\end{aligned}\right\} \tag{8}$$

The result is expressed in series form as follows when evaluated at $t = 4$:

$$\left.\begin{aligned}
y_{n+4}-4y_{n+1}+3y_n &= h^2\left[\beta_0(x)f_n+\beta_1(x)f_{n+1}+\beta_2(x)f_{n+2}+\beta_3(x)f_{n+3}+\beta_4(x)f_{n+4}\right],\\
\beta_0 &= \frac{9}{40}h^2+\frac{19}{2016}h^2u^2+\frac{83}{302400}h^2u^4+\frac{1}{197120}h^2u^6+O(u^8),\\
\beta_1 &= -\frac{83}{30}h^2+\frac{37}{1260}h^2u^2+\frac{29}{37800}h^2u^4+\frac{29}{3326400}h^2u^6+O(u^8),\\
\beta_2 &= \frac{37}{20}h^2+\frac{53}{1680}h^2u^2+\frac{11}{16800}h^2u^4-\frac{19}{4435200}h^2u^6+O(u^8),\\
\beta_3 &= \frac{11}{10}h^2-\frac{4}{315}h^2u^2-\frac{1}{9450}h^2u^4+\frac{1}{69300}h^2u^6+O(u^8),\\
\beta_4 &= \frac{7}{120}h^2+\frac{11}{10080}h^2u^2-\frac{17}{302400}h^2u^4-\frac{173}{26611200}h^2u^6+O(u^8)
\end{aligned}\right\} \tag{9}$$

The block methods are derived by evaluating the first derivative ofEquation (5) in order to obtain additional equations needed to couple with (7), (8) and (9), which gives:

$$\left.\begin{aligned}
\alpha_0' &= \frac{-1}{h}\\
\alpha_1' &= \frac{1}{h}\\
\beta_0'(t) &= \frac{1}{24}\Bigg(\frac{-96\sin(h\varpi)+144\sin(2h\varpi)-96\sin(3h\varpi)+24\sin(4h\varpi)-111h^2\varpi^2\sin(h\varpi)+96h^2\varpi^2\sin(2h\varpi)}{\varpi^2(14\sin(h\varpi)-14\sin(2h\varpi)+6\sin(3h\varpi)-\sin(4h\varpi))}\\
&+\frac{-27h^2\varpi^2\sin(3h\varpi)+312h^2t\varpi^2\sin(h\varpi)-264h^2t\varpi^2\sin(2h\varpi)+72h^2t\varpi^2\sin(3h\varpi)-150h^2t^2\varpi^2\sinh\varpi)}{\varpi^2(14\sin(h\varpi)-14\sin(2h\varpi)+6\sin(3h\varpi)-\sin(4h\varpi))}\\
&+\frac{20h^2t^3\varpi^2\sin(h\varpi)+120h^2t^2\varpi^2\sin(2h\varpi)-30h^2t^2\varpi^2\sin(3h\varpi)-16h^2t^3\varpi^2\sin(2h\varpi)+4h^2t^3\varpi^2\sin(3h\varpi)}{\varpi^2(14\sin(h\varpi)-14\sin(2h\varpi)+6\sin(3h\varpi)-\sin(4h\varpi))}\Bigg)\\
\beta_1'(t) &= \frac{-1}{24}\Bigg(\frac{-168\sin(h\varpi)+336\sin(2h\varpi)-264\sin(3h\varpi)+72\sin(4h\varpi)+34h^2\varpi^2\sin(h\varpi)-74h^2\varpi^2\sin(2h\varpi)}{\varpi^2(14\sin(h\varpi)-14\sin(2h\varpi)+6\sin(3h\varpi)-\sin(4h\varpi))}\\
&+\frac{74h^2\varpi^2\sin(3h\varpi)-27h^2\varpi^2\sin(4h\varpi)+144h^2t\varpi^2\sin(2h\varpi)-192h^2t\varpi^2\sin(3h\varpi)+72h^2t\varpi^2\sin(4h\varpi)}{\varpi^2(14\sin(h\varpi)-14\sin(2h\varpi)+6\sin(3h\varpi)-\sin(4h\varpi))}\\
&+\frac{-120h^2t^2\varpi^2\sin(h\varpi)+24h^2t^3\varpi^2\sin(h\varpi)+12h^2t^2\varpi^2\sin(2h\varpi)+72h^2t^2\varpi^2\sin(3h\varpi)-8h^2t^3\varpi^2\sin(2h\varpi)}{\varpi^2(14\sin(h\varpi)-14\sin(2h\varpi)+6\sin(3h\varpi)-\sin(4h\varpi))}\\
&+\frac{-30h^2t^2\varpi^2\sin(4h\varpi)-8h^2t^3\varpi^2\sin(3h\varpi)+4h^2t^3\varpi^2\sin(4h\varpi)}{\varpi^2(14\sin(h\varpi)-14\sin(2h\varpi)+6\sin(3h\varpi)-\sin(4h\varpi))}\Bigg)\\
\beta_2'(t) &= \frac{1}{12}\Bigg(\frac{36\sin(h\varpi)+72\sin(2h\varpi)-108\sin(3h\varpi)+36\sin(4h\varpi)-29h^2\varpi^2\sin(h\varpi)+14h^2\varpi^2\sin(2h\varpi)}{\varpi^2(14\sin(h\varpi)-14\sin(2h\varpi)+6\sin(3h\varpi)-\sin(4h\varpi))}\\
&+\frac{15h^2\varpi^2\sin(3h\varpi)-11h^2\varpi^2\sin(4h\varpi)+72h^2t\varpi^2\sin(h\varpi)-72h^2t\varpi^2\sin(3h\varpi)+36h^2t\varpi^2\sin(4h\varpi)}{\varpi^2(14\sin(h\varpi)-14\sin(2h\varpi)+6\sin(3h\varpi)-\sin(4h\varpi))}\\
&+\frac{-24h^2t^2\varpi^2\sin(h\varpi)+4h^2t^3\varpi^2\sin(h\varpi)-48h^2t^2\varpi^2\sin(2h\varpi)+72h^2t^2\varpi^2\sin(3h\varpi)+8h^2t^3\varpi^2\sin(2h\varpi)}{\varpi^2(14\sin(h\varpi)-14\sin(2h\varpi)+6\sin(3h\varpi)-\sin(4h\varpi))}\\
&+\frac{-24h^2t^2\varpi^2\sin(4h\varpi)-12h^2t^3\varpi^2\sin(3h\varpi)+4h^2t^3\varpi^2\sin(4h\varpi)}{\varpi^2(14\sin(h\varpi)-14\sin(2h\varpi)+6\sin(3h\varpi)-\sin(4h\varpi))}\Bigg)\\
\beta_3'(t) &= \frac{-1}{24}\Bigg(\frac{264\sin(h\varpi)-144\sin(2h\varpi)-24\sin(3h\varpi)+24\sin(4h\varpi)-46h^2\varpi^2\sin(h\varpi)+46h^2\varpi^2\sin(2h\varpi)}{\varpi^2(14\sin(h\varpi)-14\sin(2h\varpi)+6\sin(3h\varpi)-\sin(4h\varpi))}\\
&+\frac{-6h^2\varpi^2\sin(3h\varpi)-7h^2\varpi^2\sin(4h\varpi)+192h^2t\varpi^2\sin(h\varpi)-144h^2t\varpi^2\sin(2h\varpi)+24h^2t\varpi^2\sin(4h\varpi)}{\varpi^2(14\sin(h\varpi)-14\sin(2h\varpi)+6\sin(3h\varpi)-\sin(4h\varpi))}\\
&+\frac{-168h^2t^2\varpi^2\sin(h\varpi)+24h^2t^3\varpi^2\sin(h\varpi)+84h^2t^2\varpi^2\sin(2h\varpi)+24h^2t^2\varpi^2\sin(3h\varpi)-8h^2t^3\varpi^2\sin(2h\varpi)}{\varpi^2(14\sin(h\varpi)-14\sin(2h\varpi)+6\sin(3h\varpi)-\sin(4h\varpi))}\\
&+\frac{-18h^2t^2\varpi^2\sin(4h\varpi)-8h^2t^3\varpi^2\sin(3h\varpi)+4h^2t^3\varpi^2\sin(4h\varpi)}{\varpi^2(14\sin(h\varpi)-14\sin(2h\varpi)+6\sin(3h\varpi)-\sin(4h\varpi))}\Bigg)\\
\beta_4'(t) &= \frac{1}{24}\Bigg(\frac{120\sin(h\varpi)-96\sin(2h\varpi)+24\sin(3h\varpi)-11h^2\varpi^2\sin(h\varpi)+16h^2\varpi^2\sin(2h\varpi)-7h^2\varpi^2\sin(3h\varpi)}{\varpi^2(14\sin(h\varpi)-14\sin(2h\varpi)+6\sin(3h\varpi)-\sin(4h\varpi))}\\
&+\frac{72h^2t\varpi^2\sin(h\varpi)-72h^2t\varpi^2\sin(2h\varpi)+24h^2t\varpi_2\sin(3h\varpi)-90h^2t^2\varpi^2\sin(h\varpi)+20h^2t^3\varpi^2\sin(h\varpi)}{\varpi^2(14\sin(h\varpi)-14\sin(2h\varpi)+6\sin(3h\varpi)-\sin(4h\varpi))}\\
&+\frac{72h^2t^2\varpi^2\sin(2h\varpi)-18h^2t^2\varpi^2\sin(3h\varpi)-16h^2t^3\varpi^2\sin(2h\varpi)+4h^2t^3\varpi^2\sin3(h\varpi)}{\varpi^2(14\sin(h\varpi)-14\sin(2h\varpi)+6\sin(3h\varpi)-\sin(4h\varpi))}\Bigg)
\end{aligned}\right\} \tag{10}$$

Evaluating the first derivative at all points, and the results are expressed in the series form

$$\left.\begin{aligned}
hy_n'-y_{n+1}+y_n &= h^2\left[\beta_0(x)f_n+\beta_1(x)f_{n+1}+\beta_2(x)f_{n+2}+\beta_3(x)f_{n+3}+\beta_4(x)f_{n+4}\right],\\
\beta_0' &= -\frac{367}{1440}h^2-\frac{199}{24192}h^2u^2-\frac{1543}{362880}h^2u^4-\frac{281}{9123840}h^2u^6+O(u^8)\\
\beta_1' &= \frac{3}{8}h^2-\frac{337}{15120}h^2u^2-\frac{457}{453600}h^2u^4-\frac{289}{4435200}h^2u^6+O(u^8)\\
\beta_2' &= \frac{47}{240}h^2-\frac{353}{20160}h^2u^2-\frac{19}{40320}h^2u^4-\frac{569}{53222400}h^2u^6+O(u^8)\\
\beta_3' &= -\frac{29}{360}h^2+\frac{1}{945}h^2u^2-\frac{43}{113400}h^2u^4-\frac{127}{2494800}h^2u^6+O(u^8)\\
\beta_4' &= \frac{7}{480}h^2+\frac{289}{120960}h^2u^2+\frac{139}{518400}h^2u^4+\frac{2899}{106444800}h^2u^6+O(u^8)
\end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned}
hy_{n+1}'-y_{n+1}+y_n &= h^2\left[\beta_0(x)f_n+\beta_1(x)f_{n+1}+\beta_2(x)f_{n+2}+\beta_3(x)f_{n+3}+\beta_4(x)f_{n+4}\right],\\
\beta_0' &= \frac{3}{32}h^2+\frac{731}{120960}h^2u^2+\frac{1439}{3628800}h^2u^4+\frac{157}{5068800}h^2u^6+O(u^8)\\
\beta_1' &= -\frac{47}{90}h^2+\frac{97}{6048}h^2u^2+\frac{853}{907200}h^2u^4+\frac{5317}{79833600}h^2u^6+O(u^8)\\
\beta_2' &= -\frac{41}{240}h^2+\frac{239}{20160}h^2u^2+\frac{89}{201600}h^2u^4+\frac{743}{53222400}h^2u^6+O(u^8)\\
\beta_3' &= \frac{1}{15}h^2+\frac{1}{4320}h^2u^2+\frac{319}{907200}h^2u^4+\frac{1277}{26611200}h^2u^6+O(u^8)\\
\beta_4' &= -\frac{17}{1440}h^2-\frac{253}{120960}h^2u^2-\frac{181}{725760}h^2u^4-\frac{1681}{63866880}h^2u^6+O(u^8)
\end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned}
hy'_{n+2} - y_{n+1} + y_n &= h^2\left[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3} + \beta_4(x)f_{n+4}\right],\\
\beta'_0 &= \frac{90}{1440}h^2 + \frac{1}{640}h^2u^2 - \frac{439}{3628800}h^2u^4 - \frac{7211}{319334400}h^2u^6 + O(u^8)\\
\beta'_1 &= -\frac{361}{360}h^2 + \frac{29}{5040}h^2u^2 - \frac{17}{90720}h^2u^4 - \frac{1817}{39916800}h^2u^6 + O(u^8)\\
\beta'_2 &= \frac{37}{80}h^2 + \frac{53}{6720}h^2u^2 + \frac{11}{67200}h^2u^4 - \frac{119}{17740800}h^2u^6 + O(u^8)\\
\beta'_3 &= -\frac{13}{360}h^2 - \frac{1}{210}h^2u^2 - \frac{23}{56700}h^2u^4 - \frac{1}{22680}h^2u^6 + O(u^8)\\
\beta'_4 &= \frac{1}{288}h^2 + \frac{43}{40320}h^2u^2 + \frac{91}{518400}h^2u^4 + \frac{7097}{319334400}h^2u^6 + O(u^8)
\end{aligned}\right\}$$

(13)

$$\left.\begin{aligned}
hy'_{(n+3)} - y_{n+1} + y_n &= h^2\left[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3} + \beta_4(x)f_{n+4}\right],\\
\beta'_0 &= -\frac{119}{1440}h^2 - \frac{571}{120960}h^2u^2 - \frac{1103}{3628800}h^2u^4 - \frac{8291}{319334400}h^2u^6 + O(u^8)\\
\beta'_1 &= -\frac{9}{10}h^2 + \frac{65}{6048}h^2u^2 + \frac{517}{907200}h^2u^4 + \frac{59}{1267200}h^2u^6 + O(u^8)\\
\beta'_2 &= \frac{263}{240}h^2 + \frac{79}{20160}h^2u^2 - \frac{23}{201600}h^2u^4 - \frac{857}{53222400}h^2u^6 + O(u^8)\\
\beta'_3 &= \frac{4}{9}h^2 + \frac{167}{30240}h^2u^2 + \frac{131}{181440}h^2u^4 + \frac{5431}{79833600}h^2u^6 + O(u^8)\\
\beta'_4 &= -\frac{11}{480}h^2 - \frac{59}{17280}h^2u^2 - \frac{1241}{3628800}h^2u^4 - \frac{667}{21288960}h^2u^6 + O(u^8)
\end{aligned}\right\}$$

(14)

$$\left.\begin{aligned}
hy'_{(n+4)} - y_{n+1} + y_n &= h^2\left[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3} + \beta_4(x)f_{n+4}\right],\\
\beta'_0 &= \frac{9}{160}h^2 + \frac{29}{120960}h^2u^2 - \frac{31}{145152}h^2u^4 - \frac{89}{3225600}h^2u^6 + O(u^8)\\
\beta'_1 &= -\frac{377}{360}h^2 + \frac{5}{432}h^2u^2 - \frac{73}{453600}h^2u^4 - \frac{2089}{39916800}h^2u^6 + O(u^8)\\
\beta'_2 &= \frac{35}{48}h^2 + \frac{671}{20160}h^2u^2 + \frac{23}{28800}h^2u^4 + \frac{13}{1520640}h^2u^6 + O(u^8)\\
\beta'_3 &= \frac{161}{120}h^2 - \frac{31}{945}h^2u^2 - \frac{139}{113400}h^2u^4 - \frac{53}{831600}h^2u^6 + O(u^8)\\
\beta'_4 &= \frac{469}{1440}h^2 + \frac{1313}{120960}h^2u^2 + \frac{1741}{3628800}h^2u^4 + \frac{9721}{319334400}h^2u^6 + O(u^8)
\end{aligned}\right\}$$

(15)

Each of the coefficients in equations (7)-(9) is in trigonometric form. To avoid heavy cancellation that may occur as $u \to 0$.

Basic Properties of the Block Method

Order and Error Constant

The technique used by Enoch and Alakofa (2024) in finding the order of a method is also adopted in establishing the order of this new developed block method. With this, the proposed block method is of order $p = 5$ and error constants given by the vector

$$C_7 = \left[\frac{1}{240}, \frac{1}{120}, \frac{1}{120}, \frac{-191}{720}\right]^T$$

Zero Stability of the Block

Definition: A block method is said to be zero stable if as $h \to 0$, the roots $r_j, j = 1(1)k$ of the first characteristic polynomials $\rho(r) = 0$ that is

$$\rho(r) = det\left[\sum A^{(0)} R^{k-1}\right] = 0$$

satisfying $|R| \leq 1$, must be simple. $\rho(z) = det[zA^{(0)} - A^{(i)} = 0]$

$$\left[z\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\end{pmatrix}\right] = 0$$

$$\begin{bmatrix} z & 0 & 0 & -1\\ 0 & z & 0 & -1\\ 0 & 0 & z & -1\\ 0 & 0 & 0 & z-1\end{bmatrix} = 0$$

since $| z = 1,0,0,0 | \leq 1$. Therefore, the block method is zero stable

Consistency

Our new block method is consistent because its order is greater than 1.

Convergence

Theorem 3.1: According to Lambert (1973), a linear multistep method is convergent if and only if it is both consistent and zero stable. Thus our block method is convergent since it is zero stable and consistent.

Numerical Experiments

This section examines the performance of the new method by applying it to some second order system of equations. The test problems outcomes are displayed in tabular form.

Error= |exact solution – computed solution|

Test Problem 1

$$\begin{aligned}
y''_1 &= \frac{-y_1}{\sqrt{y_1^2 + y_2{}^2}}, y_1(0) = 1, y'_1(0) = 0, h = 0.01\\
y''_2 &= \frac{-y_2}{\sqrt{y_1^2 + y_2{}^2}}, y_2(0) = 1, y'_2(0) = 1,
\end{aligned}$$

Theoretical Solution: $y_1 = \cos x, y_2 = \sin x$

source: Abdelrahim, R. & Omar, Z. (2016).

Table 1: Exact computed solutions of the new method for solving y_1 in Test Problem 1

X	Exact solution of y_1	Computed solution of y_1	Error in y_1	Error in Abdelrahim, R. & Omar, Z. (2016)
0.1	0.995004165278025710	0.995004165424475890	1.464501853E(-10)	
0.2	0.980066577841241630	0.980066578263853460	4.226118344E(-10)	4.208611E(-11)
0.3	0.955336489125605980	0.955336489629699970	5.040939888E(-10)	-
0.4	0.921060994002885100	0.921060994660191420	6.573063205E(-10)	2.945689E(-10)
0.5	0.877582561890372760	0.877582562620329190	7.299564286E(-10)	-
0.6	0.825335614909678330	0.825335615503742240	5.940639092E(-10)	7.596528E(-10)
0.7	0.764842187284488380	0.764842187740586540	4.560981592 E(-10)	-
0.8	0.696706709347165390	0.696706709529967600	1.828022178E(-10)	1.178604E(-9)
0.9	0.621609968270664390	0.621609968134783640	1.358807511E(-10)	-
1.0	0.540302305868139770	0.540302305363824840	5.043149232E(-10)	1.180295E(-9)

Table 2: Exact and computed solutions of the new method for solving y_2 in Test Problem 1

X	Exact solution of y_2	Computed solution of y_2	Error in y_2	Error in Abdelrahim, R. & Omar, Z. (2016)
0.1	0.099833416646828155	0.09983341664532585	1.502298286E(-12)	
0.2	0.19866933079506122	0.19866933082374291	2.868169591E(-11)	3.186932E(-10)
0.3	0.29552020666133955	0.29552020706007930	4.466838011E(-11)	-
0.4	0.38941834230865052	0.38941834237389195	6.524142337E(-11)	1.082778E(-9)
0.5	0.47942553860420301	0.47942553875745147	1.532484695E(-10)	-
0.6	0.56464247339503537	0.56464247364675935	2.517239750E(-10)	1.773150E(-9)
0.7	0.64421768723769113	0.64421768754962183	3.119307035E(-10)	-
0.8	0.71735609089952279	0.71735609122482724	3.253044500E(-10)	1.862680E(-9)
0.9	0.78332690962748341	0.78332690994777632	3.202929033E(-10)	-
1.0	0.84147098480789650	0.84147098511970519	3.118086900E(-10)	1.081632E(-9)

Test Problem 2: (Periodic Problem) Vande Vyver

$$y_1'' + y = 0.001\cos(x), y_1(0) = 1, y_1'(0) = 0, h =$$

$$y_2'' + y = 0.001\sin(x), y_2(0) = 0, y_2'(0) = 0.999$$

Theoretical Solution:

$y_1(x) = \cos(x) + 0.0005x\sin(x)$,

$y_2(x) = \sin(x) - 0.0005x\cos(x)$.

source: Kayode et. al (2021)

Table 3: Exact nd computed solutions of the new method for solving y_1 in Test Problem 2

X	Exact	Computed	Error in y_1	Error in Kayode et. al (2021)
0.001	0.999999500500042	0.999999500500042	0	4.98251761E (-07)
0.002	0.999998002000665	0.999998002000666	9.992007222E (-16)	1.99750245E (-06)
0.003	0.999995504503368	0.999995504503371	2.886579864E (-15)	4.49460223E (-06)
0.004	0.999992008010645	0.999992008010650	5.107025913E (-15)	7.99469593E (-06)
0.005	0.999987512525990	0.999987512526005	1.576516695E (-14)	1.24926327E (-05)
0.006	0.999982018053892	0.999982018053925	3.264055692E (-14)	1.79955544E (-05)
0.007	0.999975524599842	0.999975524599895	5.306866058E (-14)	2.44963093E (-05)
0.008	0.999968032170325	0.999968032170401	7.593925488E (-14)	3.20040361E (-05)
0.009	0.999959540772827	0.999959540772948	1.210143097E (-13)	4.05095821E (-05)
0.01	0.999950050415832	0.999950050416012	1.800781746E (-13)	5.00240831E (-05)

Table 4: Exact and computed solutions of the new method for solving y_2 in Test Problem 2

X	Exact	Computed	Error in y_2	Error in Kayode et. al (2021)
0.001	0.000994999833583	0.000994999834367	4.499999217E (-06)	4.98251761E (-07)
0.002	0.001989998668667	0.001989998673990	8.999994677E (-06)	1.99750245E (-06)
0.003	0.002984995506752	0.002984995524214	1.349998254E (-05)	4.49460223E (-06)
0.004	0.003979989349342	0.003979989391042	1.799995830E (-05)	7.99469593E (-06)
0.005	0.004974979197943	0.004974979279097	2.249991885E (-05)	1.24926327E (-05)
0.006	0.005969964054065	0.005969964195402	2.699985866E (-05)	1.79955544E (-05)

0.007	0.006996442919223	0.006964943147947	3.149977128E (-05)	2.44963093E (-05)
0.008	0.007995914794939	0.007959915142753	3.599965219E (-05)	3.20040361E (-05)
0.009	0.008995378682741	0.008954879181453	4.049950129E (-05)	4.05095821E (-05)
0.01	0.009994833584165	0.009949834271476	4.499931269E (-05)	5.00240831E (-05)

Discussion of Result and Conclusion

The numerical results produced from the analyzed problems demonstrate high efficiency and effectiveness in their performance. From the results generated, it is worthy of note the favorable performance of the proposed method when compared with existing methods in the literature. Tables 1 and 2 presents the computational solution of test problem 1 (systems of equation), the results compete favorably well with that of Abdelrahim and Omar (2016). in terms of errors. Also, Tables 3 and 4 shows the computational solution of test problem 2 (VandeVyver), the results outperform that of Kayode et. al (2021). Utilizing collocation and interpolation techniques, a new class of continuous second derivative block methods for solving ODEs is constructed. This innovative approach combines polynomial and trigonometric functions, implemented through code written in MATLAB, to develop an approximation solution. The resulting block techniques exhibit continuous coefficients and possess key properties of consistency, zero stability, and convergence. These characteristics contribute to a robust methodology that ensures reliability in solving ODEs, presenting a promising avenue for further exploration and application in this field

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